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The massive Thirring model connection

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Abstract. The equations of the massive Thirring model are shown to have a non-trivial prolongation structure and to be related to an SL(2, R) connection form. From this an inverse scattering problem is determined and an infinite set of conservation laws constructed.

1. The prolongation structure

In characteristic coordinates the equations of the classical massive Thirring model (Thirring 1958, Coleman 1975) take the form

$$\mathrm{i}\partial_X\phi_1 = m\phi_2 + g\bar{\phi}_2\phi_2\phi_1 \tag{1.1}$$

$$\mathrm{i}\partial_T\phi_2 = m\phi_1 + g\bar{\phi}_1\phi_1\phi_2 \tag{1.2}$$

together with their complex conjugates. These equations can be represented by the closed ideal of differential forms generated by the four two-forms $\alpha_1, \alpha_2, \bar{\alpha}_1, \bar{\alpha}_2$ defined to be

$$\alpha_1 = \mathrm{d}\phi_1 \wedge \mathrm{d}T + \mathrm{i}(m\phi_2 + g\bar{\phi}_2\phi_2\phi_1)\,\mathrm{d}X \wedge \mathrm{d}T \tag{1.3}$$

$$\alpha_2 = \mathrm{d}\phi_2 \wedge \mathrm{d}X - \mathrm{i}(m\phi_1 + g\bar{\phi}_1\phi_1\phi_2)\,\mathrm{d}X \wedge \mathrm{d}T \tag{1.4}$$

$$\bar{\alpha}_1 = \mathrm{d}\bar{\phi}_1 \wedge \mathrm{d}T - \mathrm{i}(m\bar{\phi}_2 + g\bar{\phi}_2\phi_2\bar{\phi}_1)\,\mathrm{d}X \wedge \mathrm{d}T \tag{1.5}$$

$$\bar{\alpha}_2 = \mathrm{d}\bar{\phi}_2 \wedge \mathrm{d}X + \mathrm{i}(m\bar{\phi}_1 + g\bar{\phi}_1\phi_1\bar{\phi}_2)\,\mathrm{d}X \wedge \mathrm{d}T. \tag{1.6}$$

Using the techniques of Wahquist and Estabrook (1975) we can show that the one-form Ω defined by

$$\Omega = \mathrm{d}\zeta + (X_0 + X_1\phi_2 + X_2\bar{\phi}_2 + X_3\bar{\phi}_2\phi_2)\,\mathrm{d}X + (X_4 + X_5\phi_1 + X_6\bar{\phi}_1 + X_7\bar{\phi}_1\phi_1)\,\mathrm{d}T \tag{1.7}$$

is a prolongation one-form for the ideal (1.3)–(1.6) provided the generators X_i obey the following commutation relations:

$$[X_0, X_4] = 0 \qquad [X_0, X_5] = \mathrm{i}mX_1 \qquad [X_0, X_6] = -\mathrm{i}mX_2 \qquad [X_0, X_7] = 0 \tag{1.8}$$

$$[X_1, X_4] = -\mathrm{i}mX_5 \quad [X_1, X_5] = 0 \qquad [X_1, X_6] = -\mathrm{i}m(X_3 + X_7) \quad [X_1, X_7] = \mathrm{i}gX_1 \tag{1.9}$$

$$[X_2, X_4] = \mathrm{i}mX_6 \quad [X_2, X_5] = \mathrm{i}m(X_3 + X_7) \quad [X_2, X_6] = 0 \qquad [X_2, X_7] = -\mathrm{i}gX_2 \tag{1.10}$$

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$$[X_3, X_4] = 0 \qquad [X_3, X_5] = -\mathrm{i}gX_5 \qquad [X_3, X_6] = \mathrm{i}gX_6$$
$$[X_3, X_7] = 0. \tag{1.11}$$

We can simplify these relations if we make the additional *ansatz* that

$$X_4 = aX_0, \qquad X_5 = bX_1, \qquad X_6 = cX_2, \qquad X_7 = dX_0, \qquad X_3 = eX_0, \tag{1.12}$$

then the relations (1.8)–(1.11) imply that

$$[X_0, X_1] = \frac{\mathrm{i}m}{b}X_1 = \frac{\mathrm{i}mb}{a}X_1 = -\frac{\mathrm{i}g}{d}X_1 = -\frac{\mathrm{i}g}{e}X_1 \tag{1.13}$$

$$[X_0, X_2] = -\frac{\mathrm{i}m}{c}X_2 = -\frac{\mathrm{i}mc}{a}X_2 = \frac{\mathrm{i}g}{d}X_2 = \frac{\mathrm{i}g}{e}X_2 \tag{1.14}$$

$$[X_1, X_2] = -\frac{\mathrm{i}m(d+e)}{c}X_0 = -\frac{\mathrm{i}m(d+e)}{b}X_0 \tag{1.15}$$

and we see that we must select a, b, c, d and e so that

$$\mathrm{i}m/b = \mathrm{i}mb/a = -\mathrm{i}g/d = -\mathrm{i}g/e \tag{1.16}$$

$$-\mathrm{i}m/c = -\mathrm{i}mc/a = \mathrm{i}g/d = \mathrm{i}g/e \tag{1.17}$$

$$-\mathrm{i}m(d+e)/c = -\mathrm{i}m(d+e)/b. \tag{1.18}$$

These relations have the general solution

$$a = b^2, \qquad c = b, \qquad d = e = -gb/m, \qquad \text{with } b \text{ arbitrary.} \tag{1.19}$$

If we define $b = \lambda^{-2}$ then the commutation relations of X_0, X_1, X_2 take the form

$$[X_0, X_1] = \mathrm{i}m\lambda^2 X_1 \qquad [X_0, X_2] = -\mathrm{i}m\lambda^2 X_2 \qquad [X_1, X_2] = 2\mathrm{i}gX_0. \tag{1.20}$$

If we define Y_0, Y_{+1}, Y_{-1}, by the identifications

$$X_0 = \mathrm{i}m\lambda^2 Y_0 \qquad X_1 = (gm)^{1/2}\lambda \mathrm{i} Y_{+1} \qquad X_2 = (gm)^{1/2}\lambda \mathrm{i} Y_{-1} \tag{1.21}$$

then the algebra of the Y_0, $Y_{\pm 1}$ is

$$[Y_0, Y_{+1}] = Y_{+1} \qquad [Y_0, Y_{-1}] \qquad [Y_{+1}, Y_{-1}] = +2Y_0 \tag{1.22}$$

which is the algebra of SL(2, R). A one-dimensional representation of SL(2, R) is given by

$$Y_0 = \zeta\partial/\partial\zeta \qquad Y_{+1} = \zeta^2\partial/\partial\zeta \qquad Y_{-1} = -\partial/\partial\zeta \tag{1.23}$$

and yields the following representation of the generators X_i:

$$X_0 = \mathrm{i}m\lambda^2\zeta\partial/\partial\zeta \qquad X_1 = \mathrm{i}(gm)^{1/2}\lambda\zeta^2\partial/\partial\zeta \qquad X_2 = -\mathrm{i}(gm)^{1/2}\lambda\,\partial/\partial\zeta$$
$$X_3 = -\mathrm{i}g\zeta\partial/\partial\zeta \qquad X_4 = \mathrm{i}m\lambda^{-2}\zeta\partial/\partial\zeta \qquad X_5 = \mathrm{i}(gm)^{1/2}\lambda^{-1}\zeta\partial/\partial\zeta$$
$$X_0 = -\mathrm{i}(gm)^{1/2}\lambda^{-1}\partial/\partial\zeta \quad X_7 = -\mathrm{i}g\zeta\partial/\partial\zeta.$$

This gives the prolongation form

$$\Omega = \mathrm{d}\zeta + \mathrm{i}(m\lambda^2\zeta + \lambda(mg)^{1/2}\zeta^2\phi_2 - (mg)^{1/2}\lambda\bar{\phi}_2 - g\zeta\bar{\phi}_2\phi_2)\,\mathrm{d}X \tag{1.27}$$
$$+\mathrm{i}(m\lambda^{-2}\zeta + (mg)^{1/2}\lambda^{-1}\zeta^2\phi_1 - (mg)^{1/2}\lambda^{-1}\bar{\phi}_1 - g\zeta\bar{\phi}_1\phi_1)\,\mathrm{d}T. \tag{1.28}$$

From this we see that the Cartan-Ehresman connection (Hermann 1976)

$$\omega = \omega_0 + \omega_1\zeta + \omega_2\zeta^2 \tag{1.29}$$

where

$$\omega_0 = -\mathrm{i}(mg)^{1/2}(\lambda\bar{\phi}_2\,\mathrm{d}X + \lambda^{-1}\bar{\phi}_1\,\mathrm{d}T) \tag{1.30}$$

$$\omega_1 = \mathrm{i}[(m\lambda^2 - g\bar{\phi}_2\phi_2)\,\mathrm{d}X + (m\lambda^{-2} - g\bar{\phi}_1\phi_1)\,\mathrm{d}T] \tag{1.31}$$

$$\omega_2 = \mathrm{i}(mg)^{1/2}(\lambda\phi_2\,\mathrm{d}X + \lambda^{-1}\phi_1\,\mathrm{d}T) \tag{1.32}$$

has the property that its curvature forms

$$\Omega_0 = \mathrm{d}\omega_0 - \omega_0 \wedge \omega_1 = \lambda\bar{\alpha}_2 + \lambda^{-1}\bar{\alpha}_1 \tag{1.33}$$

$$\Omega_1 = \mathrm{d}\omega_1 - 2\omega_0 \wedge \omega_2 = -\mathrm{i}g(\phi_2\bar{\alpha}_2 + \bar{\phi}_2\alpha_2 + \phi_1\bar{\alpha}_1 + \bar{\phi}_1\alpha_1) \tag{1.34}$$

$$\Omega_2 = \mathrm{d}\omega_2 + \omega_2 \wedge \omega_1 = \lambda\alpha_2 + \lambda^{-1}\alpha_1 \tag{1.35}$$

lie in the ring of forms generated by α_1, α_2, $\bar{\alpha}_1$, $\bar{\alpha}_2$. Thus the equations of the massive Thirring model can be associated with the vanishing of an SL(2, R) connection in an analogous way to the non-linear Schrödinger equation (Corones 1977, Morris 1977).

Sectioning Ω onto a solution manifold of (1.1)–(1.2) gives us a non-linear scattering problem in the Ricatti form,

$$\zeta_X = -\mathrm{i}[m\lambda^2\zeta + \lambda(mg)^{1/2}\zeta^2\phi_2 - (mg)^{1/2}\lambda\bar{\phi}_2 - g\bar{\phi}_2\phi_2\zeta] \tag{1.36}$$

$$\zeta_T = -\mathrm{i}[m\lambda^{-2}\zeta + \lambda^{-1}(mg)^{1/2}\zeta^2\phi_1 - (mg)^{1/2}\lambda^{-1}\bar{\phi}_1 - g\bar{\phi}_1\phi_1\zeta]. \tag{1.37}$$

If we take a two-dimensional representation for the Y_0, $Y_{\pm1}$,

$$Y_0 = \tfrac{1}{2}(\zeta^2\partial/\partial\zeta^2 - \zeta^1\partial/\partial\zeta^1) \qquad Y_{+1} = \zeta^2\partial/\partial\zeta^1 \qquad Y_{-1} = \zeta^1\partial/\partial\zeta^2 \tag{1.38}$$

then we obtain a linear inverse scattering problem of the form

$$\zeta^1_X = \tfrac{1}{2}\mathrm{i}(m\lambda^2 - g\bar{\phi}_2\phi_2)\zeta^1 - \mathrm{i}(mg)^{1/2}\lambda\phi_2\zeta^2 \tag{1.39}$$

$$\zeta^2_X = -\tfrac{1}{2}\mathrm{i}(m\lambda^2 - g\bar{\phi}_2\phi_2]\zeta^2 - \mathrm{i}(mg)^{1/2}\lambda\bar{\phi}_2\zeta^1 \tag{1.40}$$

$$\zeta^1_T = \tfrac{1}{2}\mathrm{i}(m\lambda^{-2} - g\bar{\phi}_1\phi_1)\zeta^1 - \mathrm{i}(mg)^{1/2}\lambda^{-1}\phi_1\zeta^2 \tag{1.41}$$

$$\zeta^2_T = -\tfrac{1}{2}\mathrm{i}(m\lambda^{-1} - g\bar{\phi}_1\phi_1)\zeta^2 - \mathrm{i}(mg)^{1/2}\lambda^{-1}\bar{\phi}_1\zeta^1. \tag{1.42}$$

This form has also been determined by Michaelov (1976) using different methods. The inverse scattering problem above can be solved (A C Newell 1978, private communication) and multisoliton solutions determined.

2. Conservation laws

From equations (1.36) and (1.37) one easily discoveres that

$$\mathrm{i}[\lambda(\phi_2\zeta)_T - \lambda^{-1}(\phi_1\zeta)_X] = (mg)^{1/2}(\phi_1\bar{\phi}_2 - \bar{\phi}_1\phi_2). \tag{2.1}$$

Thus if we expand ζ as a series in λ,

$$\zeta = \sum_{i=0}^{\infty} \lambda^{2i+1}\zeta_i \tag{2.2}$$

the quantity C_i defined by

$$C_i = \int \phi_2 \zeta_i \, \mathrm{d}X \tag{2.3}$$

is conserved for all i.

The first two coefficients in the series for ζ are

$$\zeta_0 = (g/m)^{1/2} \bar{\phi}_1 \qquad \zeta_1 = m^{-1}(g/m)^{1/2}(\mathrm{i}\bar{\phi}_{1T} + g\bar{\phi}_1 \phi_1 \bar{\phi}_1) \tag{2.4}$$

with the corresponding conserved quantities

$$C_0 = \int (g/m)^{1/2} \phi_2 \bar{\phi}_1 \, \mathrm{d}X$$

and

$$C_1 = \int m^{-1}(g/m)^{1/2} \phi_2 (\mathrm{i}\bar{\phi}_{1T} + g\bar{\phi}_1 \phi_1 \bar{\phi}_1) \, \mathrm{d}X. \tag{2.5}$$

The above result is more concisely expressed if we note that the potential one-form ω defined by,

$$\omega = (\lambda \phi_2 \, \mathrm{d}X + \lambda^{-1} \phi_1 \, \mathrm{d}T)\zeta - (g/m)^{1/2} \bar{\phi}_1 \phi_1 \, \mathrm{d}T = \sum_{i=0}^{\infty} \lambda^{2i+1} \omega_i \tag{2.6}$$

can easily be shown to provide a prolongation of the ideal (1.3)–(1.6) prolonged by the one-forms Ω and $\bar{\Omega}$. Thus we must have

$$\int_C \omega_i = 0 \tag{2.7}$$

where C is any curve lying in the solution manifold of the Thirring equations (1.1)–(1.2). Equations (2.6) and (2.7) should also be compared with the analogous expressions for the anticommuting Thirring model (Morris 1978).

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